Kawamata–Shokurov base point free theorem.

(X, Δ) klt pair, $L \in \mathrm{Div}\, X$ nef, $pL - (K_X + \Delta)$ nef big, $p \in \mathbb{Q}_{>0}$

$\Rightarrow |mL|$ is bpf $\forall m \gg 0$.

Lemma: (X, Δ) klt, $|D|$ lin sys, $\dim |D| > 0$, $S \in |D|$, $c := \min \left\{ t \in \mathbb{Q} \mid K_X + \Delta + tS \text{ not klt} \right\}$

Then if $c < 1 \Rightarrow \forall W$ lc centre of $(X, \Delta + cS)$: $W \subseteq \mathrm{Bs}\,|D|$

if $c \geq 1 \Rightarrow$ either $\exists W$ lc centre s.t. $W \subseteq \mathrm{Bs}\,|D|$

or S reduced, with ired conn components that are the lc centr

PF OF THM:

① $pL - (K_X + \Delta)$ is nef & big $\Rightarrow \sim \frac{1}{m}(A + E)$ for $m \gg 0$, A ample, $E \geq 0$.

$\to pL - (K_X + \Delta) - \varepsilon \cdot F$ is ample for $\varepsilon \ll 1$, $F \geq 0$ suitable.

$(X, \Delta + \varepsilon F)$ is klt for $\varepsilon \ll 1$ by alg and (X, Δ) being klt.

$\Rightarrow$ wma $pL - (K_X + \Delta)$ to be ample.

② $\mathrm{Bs}\,|mL| = \emptyset \quad \forall m \gg 0 \iff \forall d \geq 2\ \exists e \geq 1 : \mathrm{Bs}\,|d^e L| = \emptyset$
$(\iff \forall d\ \exists e_0\ \forall e \geq e_0\ \mathrm{Bs}\,|d^e L| = \emptyset)$

↯ $\exists d \geq 2\ \forall e \geq 1\ \mathrm{Bs}\,|d^e L| = \emptyset$

$\forall m$: $\mathrm{Bs}\,|d^{m+1} L| \subseteq \mathrm{Bs}\,|d^m L|$ are closed sets, X is noetherian

$\Rightarrow \mathrm{Bs}\,|d^e L|$ stabilises for $e \gg 0$. $\Rightarrow$ by replacing L by a multiple

wma $\mathrm{Bs}\,|d^e L| = \underset{\neq \emptyset}{\mathrm{Bs}\,|dL|}$ $\forall e \geq 1$, ~~and~~ $\Rightarrow$ $|dL| \neq \emptyset$

Let $D \in |dL|$, $c := \inf \left\{ t \in \mathbb{Q} \mid K_X + \Delta + tD \ \neg \text{klt} \right\}$ klt threshold

Let W be a lc centre of $(X, \Delta + cS)$ s.t. $W \cap \mathrm{Bs}\,|dL| \neq \emptyset$
(doable by lemma)

Let $q := d^e \geq p + d$. $\Rightarrow qL \sim_{\mathbb{Q}} K_X + \Delta + D + A$ for A ample $\mathbb{Q}$-Cartier.
(because $pL - (K_X + \Delta)$ is ample and L is nef), $D \in |dL|$)

Tie breaking $\Rightarrow \exists \varepsilon, \eta$: W isol lc centre of $(X, \Delta + (1-\varepsilon)cD + \eta A)$

$\Rightarrow I_W = \mathcal{J}(X, \Delta + (1-\varepsilon)cD + \eta A)$

$$qL \sim_{\mathbb{Q}} K_X + \Delta + (1-\varepsilon)cD + \eta A + \overbrace{(1-\eta)A + \varepsilon c D + (1-c)D}^{A \text{ ample, } D \text{ nef } \Rightarrow \text{ ample}}$$

Kawamata subadj: $qL|_W = K_W + \Delta_W + A_W$ where A_W ample

& Induction on dimension: (W, Δ_W) klt, $qL|_W - (K_W - \Delta_W)$ ample $\Rightarrow$
$\to$ ~~[illegible]~~ $|mL|_W|$ bpf $\forall m \gg 0$

Nadel vanishing: $H^0(X, qL) \longrightarrow H^0(W, mL|_W) \longrightarrow \underbrace{H^1(W, qL|_W)}_{0}$
$\forall m \gg 0$

$\Rightarrow |mL|_W|$ not bpf ↯

Kawamata's Rationality Theorem. (X, Δ) klt, $K_X + \Delta$ not nef,

$a \in \mathbb{Z}_{>0}$: $a \cdot (K_X + \Delta)$ Cartier, H nef big Cartier,

$r := \sup \{ t \in \mathbb{Q} \mid H + t(K_X + \Delta) \text{ nef} \}$ nef threshold $\Rightarrow r = \frac{u}{v} \in \mathbb{Q}$, $v \leq a \cdot (\dim X + 1)$

Lemma. $P(x,y) \in [x,y] \setminus \{0\}$, $\deg P \leq n$, $a \in \mathbb{Z}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$,

$\exists r \in \mathbb{R}$: $\forall x,y \gg 0$, $0 < ay - rx < \varepsilon$: $P(x,y) = 0$. $\Rightarrow r = \frac{u}{v} \in \mathbb{Q}$, $v \leq \frac{a \cdot (n+1)}{\varepsilon}$

Pf: Suppose $r \in \mathbb{R} \setminus \mathbb{Q}$ $\Rightarrow$ $\exists$ infinitely many (x,y) that are of distance

$< \frac{\varepsilon}{n+2}$ to the line $ay - rx$ $\Rightarrow$ $\exists (x',y')$ s.t. $P(x',y') = 0$ and

$0 < ay' - rx' < \frac{\varepsilon}{n+2}$ $\Rightarrow$ $(x',y'), (2x',2y'), \dots, ((n+1)x', (n+1)y')$ are

roots of P, P is of deg n $\Rightarrow$ $(y'x - x'y) \mid P$.

Repeat for an even smaller ε, $\Rightarrow$ do this $n+1$ times

$\Rightarrow$ $\deg P = n$ but $\deg P \geq n+1$ ↯

$\Rightarrow r = \frac{u}{v} \in \mathbb{Q}$. Let $j \in \mathbb{Z}$ and $ay' - rx' = \frac{aj}{v}$ for $(x', y') \in \mathbb{Z}^2$;

such a pair (x',y') exists. $\Rightarrow$ $a \cdot (y' + ku) - r(x' + akv) = \frac{aj}{v}$ $\forall k$

$\Rightarrow$ if $\frac{aj}{v} < \varepsilon$ then $ay - rx - \frac{aj}{v} \mid P$.

$\Rightarrow$ There are $\leq n$ such values of j $\Rightarrow$ $\frac{a \cdot (n+1)}{v} \geq \varepsilon$.

PF OF THM:

Cone Theorem. (X,Δ) projective klt pair. Then there are countably many extremal rays $R_i \subseteq \overline{NE}(X)$ that are $(K_X+\Delta)$-negative s.t.

a) $\overline{NE}(X) = \overline{NE}_{K_X+\Delta \geq 0} + \sum_i R_i$, $\quad R_i = \overline{NE} \cap L_i^\perp$

b) $\forall H$ ample divisor $\forall \varepsilon > 0$: $\overline{NE}(X) = \overline{NE}_{K_X+\Delta+\varepsilon H \geq 0} + \sum_{\text{fin}} R_i$

c)[PO] Every R_i is generated by a rational curve.
(Uses bend and break.)

Pf: ① $L \in \operatorname{Div} X$ nef, $\overline{NE}_{K_X+\Delta<0} \cap L^\perp \neq \emptyset \Rightarrow \exists M \in \operatorname{Div} X$ nef,
$\exists R \subseteq (\overline{NE}_{K_X+\Delta<0} \cap L^\perp) \cup \{0\}$ extremal ray s.t. $R = \overline{NE}(X) \cap M^\perp$

one point on the figure

② $\overline{NE}(X) = \overline{NE}_{K_X+\Delta \geq 0} + \sum_{\substack{L \text{ nef} \\ \dim F_L = 1}} F_L \quad (=: W)$

③ H ample, $\varepsilon > 0 \Rightarrow$ there are only finitely many extremal rays F_L s.t. $(K_X + \Delta + \varepsilon H).F_L < 0$.

② + ③: $\overline{NE}(X) = \overline{NE}_{K_X+\Delta+\varepsilon H \geq 0} + \sum_{\text{fin}} R_i \Rightarrow$ b)

The accumulation pts of $\sum R_i$ are on $(K_X+\Delta = 0) \Rightarrow$ a)

Rat Thm

Pf of ①: $H \in \operatorname{Div} X$ ample, $d \in \mathbb{N}$, $r_L(d,H) := \sup\{t \in \mathbb{Q} \mid dL + H + t\cdot(K_X+\Delta) \text{ nef}\} \in \mathbb{Q}$

If $K_X+\Delta$ is nef, the statement is trivial.

L nef $\Rightarrow r_L$ non-increasing in d.

~~reversed!~~

$\exists z \in \underbrace{\overline{NE}_{K_X+\Delta<0} \cap L^\perp}_{\neq \emptyset}$. Then $(\overbrace{dL + H + r_L(d,H)\cdot(K_X+\Delta)}^{=: M_*(dL,H)}).z \geq 0 \Leftrightarrow r_L(d,H) \leq \dfrac{H.z}{-(K_X+\Delta).z}$

Rationality $\Rightarrow$ denom of $r_L(d,H) \leq a\cdot(\dim X + 1)$, $a \in \mathbb{N}$, $a\cdot(K_X+\Delta) \in \operatorname{Div} X$

$\Rightarrow r_L(d,H)$ is constant for $d \geq d_0 \in \mathbb{N}$

by def of r_L

~~Kleiman~~?
on the bdry of the cone, ample is the int

$M(dL,H) := dL + H + r_L(d,H)\cdot(K_X+\Delta)$ nef but not ample

$\forall d \geq d_0$: $\{0\} \neq F_{M(dL,H)} \subseteq F_L := \overline{NE}(X) \cap L^\perp$, $F_{M(dL,H)} \subseteq \overline{NE}_{K_X+\Delta<0} \cup \{0\}$

Kleiman Crit

If $\dim F_{M(dL,H)} = 1 \longrightarrow M = M(dL,H)$ does the job

If $\dim \geq 2$: $H_1, \dots, H_\rho$ basis of $N^1(X)$ (fin gen, Severi) s.t. $\forall H_i$ ample

$\Rightarrow (dL + H_i + r_L(dL,H_i)(K_X+\Delta))\big|_{F_{M(dL,H)}} \not\equiv 0$ for some i

$\Rightarrow R = F_{M(dL,H)}$

$$\cancel{(dL + H + r_L(K_X+\Delta))z = \underbrace{dL.z}_{0} + H.z + r_L(K_X+\Delta)z \geq}$$

$$\cancel{\frac{H.z}{(K_X+\Delta).z} \leq + r_L}$$

Pf of ②: $\{ \overline{NE}(X) \supsetneq W \Rightarrow \exists z_0 \in \overline{NE}(X) \setminus W,\ M \in \mathrm{Div}\,X,$

$M.z_0 < 0$ and $M.(W \setminus \{0\}) > 0$. (separation by hyperplane; W is a cone)

$C := \{ D \in N^1(X) \mid D.z \geq 0\ \forall z \in \overline{NE}_{K_X+\Delta \geq 0} \}$ dual cone

C is gen'd by $K_X + \Delta$ and nef divisors ~~(WHY?)~~

$M.(W\setminus 0) > 0 \Rightarrow M \in \mathrm{int}\, C \Rightarrow M = A + p\cdot(K_X+\Delta)$ for A ample, $p \in \mathbb{Q}_{>0}$ int

$r := \sup\{ t \in \mathbb{Q} \mid A + t(K_X+\Delta) \text{ nef} \} \in \mathbb{Q}$ by Rat. Thm.

$D := A + r(K_X+\Delta)$ is nef, and not ample $\xrightarrow{\text{Kleiman}}$ $\underbrace{D^\perp \cap \overline{NE}_{K_X+\Delta<0} \neq \emptyset}_{}$ ~~WHY?~~

① $\Rightarrow \exists R$ extremal ray, $R \subseteq F_D$, ~~WHY?~~ $M.(W\setminus 0) > 0$ ~~shouldn't this be $M.R \geq 0$?~~ Yes.

$\Rightarrow (A + r(K_X+\Delta)).R = D.R = 0$ and $(A + p(K_X+\Delta)).R = M.R > 0$ ~~WHAT?~~

~~WHY?~~ int

$\Rightarrow p < r \Rightarrow M$ ample, $M.z_0 < 0$ contradicts Kleiman's Criterion ↯

$0 < M.R - D.R = (M-D).R = (p-r)\underbrace{(K_X+\Delta).R}_{<0} \Rightarrow p - r < 0 \Rightarrow p < r$

Pf of ③: Let $\{F_{L_i} \mid i\}$ be all the extremal rays f.w. $(K_X+\Delta+\varepsilon H)F_{L_i} < 0$

Wts there are only finitely many.

$\forall i$ let $z_i \in F_{L_i}$ s.t. $(K_X+\Delta).z_i = -1$ ~~(WHY do these exist?)~~

$\Rightarrow \forall i: \quad 0 < H.z_i < \varepsilon$ (Kleiman) $\quad (K_X+\Delta+\varepsilon H).z_i = (K_X+\Delta).z_i + \varepsilon H.z_i = -1 + \varepsilon \cdot H z_i > 0 \Rightarrow H z_i > \frac{1}{\varepsilon}$

Let $D_1, \ldots, D_\rho$ be a basis of $N^1(X)$ of ample Cartier divisors (Severi)

(why does such a base exist?)

$\exists m > 0 : \forall i: mH - D_i$ is ample ($\exists$ by Serre)

Kleiman's Criterion $\Rightarrow \forall i \forall j \quad (mH - D_i)z_j > 0$

$\Rightarrow D_i.z_j < mH.z_j < m\cdot\varepsilon$

$$\left.\begin{array}{l}\big(\underbrace{dL_j + D_i + r_{L_j}(d, D_i)\cdot(K_X+\Delta)}_{\text{nef divisor}}\big).z_j > 0\end{array}\right\} \text{as in ①}$$

Rat. Thm. $\Rightarrow D_i.z_j = r_{L_j}(d, D_i)$ has bounded denominator

Look at the coordinates in the basis $D_1, \ldots, D_\rho$ for z_j

$\Rightarrow$ there are only fin many z_j